Complex Properties and Chaos Synchronization of a New Chaotic Complex Nonlinear Dynamical System.

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Abstract

In this paper, we introduce a new chaotic complex nonlinear system and discuss some basic properties of this system including invariance, dissipativity, equilibria and their stability, Lyapunov exponents, chaotic behavior and chaotic attractors are studied. This work presents chaos synchronization between two identical chaotic complex system by using nonlinear control technique. This technique is applied to achieve chaos synchronization. A Lyapunov function is derived to prove that the error system is asymptotically stable. These expressions are tested numerically and excellent agreement is found.

Keywords: chaos, synchronization, chaotic system, nonlinear control, error system, Lyapunov functions, complex.

1-Introduction

Research in the area of the synchronization of dynamical systems dates back over 300 years. Huygens, most famous for his studies in optics and the construction of telescopes and clocks, was probably the first scientist who observed and described the synchronization phenomenon as early as in the 17th century. The pioneering paper on the concept of chaos synchronization was not presented until 1990. Pecora and Carroll introduced a method [1] to synchronize two identical chaotic systems with different initial conditions. Because of their works, chaos synchronization has been intensively studied in the last few years. It has been widely explored in a variety of fields including physical, chemical and ecological [2] systems and secure communications [3--5].

Chaos synchronization is a very important nonlinear phenomenon, which has been studied to date on dynamical systems described by real variables. There also exist, however, interesting cases of dynamical systems, where the main variables participating in the dynamics are complex, as for example when amplitudes of electromagnetic fields are involved. Another example is when chaos synchronization is used for communications, where doubling the number of variables may be used to increase the content and security of the transmitted information. A similar generalization of the real Lorenz system to the corresponding one with complex ODEs has been introduced to describe and simulate the physics of laser and thermal convection of liquid flows [6-10]. The electric field and atomic polarization amplitude in these systems are both complex quantities, whose real and imaginary parts can display chaotic dynamics, for more details see, e.g. [11] and references therein.

Recently, a generalization for the autonomous real nonlinear chaotic system to the complex system were introduced by Mahmoud and et al. The dynamical properties and chaotic synchronization were studied for this systems, and are shown that they are chaotic and exhibit chaotic attractors. The fixed points and their stability are studied of these complex chaotic systems [12].

In Physica A Qi and et al introduce new chaotic system [13], which is described by:

International Journal of Scientific & Engineering Research Volume 9, Issue 10, October-2018
ISSN 2229-5518

$$\begin{aligned} \dot{x} &= a(y-x)+yz, \\ \dot{y} &= cx-y-xz, \\ \dot{z} &= xy-bz, \end{aligned} \tag{1}$$

where x, y, z are real variables.

In this paper we introduce the complex version of (1), which is written as:

$$\begin{aligned} \dot{x} &= a(y-x)+yz, \\ \dot{y} &= cx-y-xz, \\ \dot{z} &= 1/2(\bar{x}y+x\bar{y})-bz, \end{aligned} \tag{2}$$

where a, b, c are positive parameters, $x = u_1 + iu_2, y = u_3 + iu_4$ are complex function, $i = \sqrt{-1}$ and z is a real function. Dots represent derivatives with respect to time and $(\bar{\dots})$ the complex conjugate function.

This paper is organized as follows: In Section 2, the dynamical properties of system (2) including invariance, dissipativity, equilibria and their stability, Lyapunov exponents, chaotic behavior and chaotic attractors are studied. Section 3 contains the study of chaos synchronization of (2) using the nonlinear control technique[14-19]. In this section, also, we calculate expressions for the control functions which are used to achieve chaos synchronization. These expressions are tested numerically and excellent agreement is found. A Lyapunov function is derived to prove that the error system is asymptotically stable. Some figures are presented to show chaos synchronization and errors. Finally, in Section 4 the main conclusions of our investigations are summarized.

2-Dynamical Behaviors of System (2).

In this section we study the basic dynamical analysis of our new system (2). The real version of (2) reads:

$$\begin{aligned} \dot{u}_1 &= a(u_3-u_1)+u_3u_5, \\ \dot{u}_2 &= a(u_4-u_2)+u_4u_5, \\ \dot{u}_3 &= cu_1-u_1u_5-u_3, \\ \dot{u}_4 &= cu_2-u_2u_5-u_4, \\ \dot{u}_5 &= u_1u_3+u_2u_4-bu_5. \end{aligned} \tag{3}$$

System (3) has the following basic dynamical properties:

(2.1) Symmetry and invariance:

Symmetry about the u_5 -axis, which is invariant for the coordinate transformation $(u_1, u_2, u_3, u_4, u_5) \rightarrow (-u_1, -u_2, -u_3, -u_4, u_5)$.

(2.2) Dissipation:

The divergence of (3) is :

$$\nabla . F = \sum_{i=1}^{5} \frac{\partial \dot{u}_i}{\partial u_i} = -2a-b-2.$$

Therefore the system (3) is dissipative for the case

$$a > -\frac{b}{2} - 1.$$

(2.3) Equilibria and their stability:

The equilibria of system (3) can be found by solving the following system of equations:

$$\begin{aligned} a(u_3 - u_1) + u_3 u_5 &= 0,\\ a(u_4 - u_2) + u_4 u_5 &= 0,\\ cu_1 - u_1 u_5 - u_3 &= 0,\\ cu_2 - u_2 u_5 - u_4 &= 0,\\ u_1 u_3 + u_2 u_4 - bu_5 &= 0. \end{aligned} \tag{4}$$

An isolated one at $E_0 = (0,0,0,0,0)$ trivial fixed point, as well as two a whole *circle* of equilibria given by the expression:

$$u_1^2 + u_2^2 = \beta, u_1^2 + u_2^2 = \beta^*,$$

where:

$$\beta = \frac{b}{4a}[(c+\alpha)^2 - a^2], \beta^* = \frac{b}{4a}[(c-\alpha)^2 - a^2], \alpha = \sqrt{-4a + a^2 + 2ac + c^2}.$$

The nontrivial fixed points can be written in the form:

$$E_{\theta_1} = (u_1, \pm\sqrt{\beta - u_1^2}, (c-\gamma)u_1, \pm(c-\gamma)\sqrt{\beta - u_1^2}, \gamma),$$
$$E_{\theta_2} = (u_1, \pm\sqrt{\beta^* - u_1^2}, (c-\gamma^*)u_1, \pm(c-\gamma^*)\sqrt{\beta^* - u_1^2}, \gamma^*),$$

where $\gamma = \frac{1}{2}(c - a + \alpha)$ and $\gamma^* = \frac{1}{2}(c - a - \alpha)$.

The eigenvalues of the corresponding linearized system at E_0 are:

$$\zeta_1 = -b, \zeta_{2,3} = \frac{1}{2}(-1 - a - \sqrt{1 - 2a + a^2 + 4ac}),$$
$$\zeta_{4,5} = \frac{1}{2}(-1 - a + \sqrt{1 - 2a + a^2 + 4ac}).$$

The equilibrium E_0 is stable if $b > 0$ and $c < 1,$ otherwise it becomes unstable.

To study the stability of E_{θ_1} we have set $u_1 = \sqrt{\beta}\cos\theta_1$, $u_2 = \sqrt{\beta}\sin\theta_1, u_3 = (c-\gamma)\sqrt{\beta}\cos\theta_1$ and $u_4 = (c-\gamma)\sqrt{\beta}\sin\theta_1$ for $\theta_1 \in [0, 2\pi]$.

We consider the Jacobian matrix of system (3) at E_{θ_1} :

$$J_{E_{\theta_1}} = \begin{pmatrix} -a & 0 & a+\gamma & 0 & (c-\gamma)\sqrt{\beta}\cos\theta_1 \\ 0 & -a & 0 & a+\gamma & (c-\gamma)\sqrt{\beta}\sin\theta_1 \\ c-\gamma & 0 & -1 & 0 & -\sqrt{\beta}\cos\theta_1 \\ 0 & c-\gamma & 0 & -1 & -\sqrt{\beta}\sin\theta_1 \\ (c-\gamma)\sqrt{\beta}\cos\theta_1 & (c-\gamma)\sqrt{\beta}\sin\theta_1 & \sqrt{\beta}\cos\theta_1 & \sqrt{\beta}\sin\theta_1 & -b \end{pmatrix}.$$

We find that their eigenvalues satisfy the characteristic polynomial:

$$\zeta(\zeta + a + 1)(\zeta^3 + L_e\zeta^2 + L_r\zeta + L_s) = 0,$$

where $L_e = 1 + a + b, L_r = \frac{b}{2}[a(a + c - \alpha) + \frac{c}{a}(c + a + \alpha)]$ and $L_s = ab(b + 2c - \alpha - 4) + bc(\alpha + c)$.

According to Routh-Hurwitz theorem the necessary and sufficient conditions for all the roots have negative real parts if and only if:

$$a > -1, L_e > 0, L_r > 0 \text{ and } L_eL_r - L_s > 0.$$

Otherwise these fixed points are unstable.

For nontrivial equilibria E_{θ_2} , their stabilities can be studied by the same way for E_{θ_1} .

(2.4) Lyapunov exponents:

System (2) in vector notation can be written as:

$$\dot{U}(t) = h(U(t);\eta), \quad (5)$$

where $U(t) = [u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)]^t$ is the state space vector, $h = [h_1, h_2, h_3, h_4, h_5]^t, \eta$ is a set of parameters and $[...]^t$ denoting transpose. The equations for small deviations δU from the trajectory $U(t)$ are:

$$\delta\dot{U}(t) = L_{ij}(U(t);\eta)\delta U, \quad i,j = 1,2,3,4,5 \quad (6)$$

where $L_{i,j} = \frac{\partial h_i}{\partial u_j}$ is the Jacobian matrix of the form:

$$L_{i,j} = \begin{pmatrix} -a & 0 & a+u_5 & 0 & u_3 \\ 0 & -a & 0 & a+u_5 & u_4 \\ c+u_5 & 0 & -1 & 0 & -u_1 \\ 0 & c+u_5 & 0 & -1 & -u_2 \\ u_3 & u_4 & u_1 & u_2 & -b \end{pmatrix}.$$

The Lyapunov exponents λ_i of the system is defined by [20]:

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log\frac{\|\delta u_i(t)\|}{\|\delta u_i(0)\|}. \quad (7)$$

To find λ_i, Eqs. (5) and (6) must be numerically solved simultaneously. Runge-Kutta method of order 4 is used to calculate λ_i.
For the case $a = 42, b = 6, c = 28$ we calculate the Lyapunov exponents as:
$\lambda_1 = 2.24, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = -61.9, \lambda_5 = -72.84.$
This means that our system (2) for this choice of a, b and c is a chaotic system since one of the Lyapunov exponents is positive.
In Figure 1a we plot the numerical calculation of the maximal Lyapunov exponent λ, with the above choice of a, b and c, which is defined as [20]:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \frac{\| \delta U(t) \|}{\| \delta U(0) \|}. \quad (8)$$

It is clear that our attractor is chaotic, since λ is positive.
Other sign of chaotic behavior of this system can be shown by plotting the separation of two nearby trajectories. Figure (1b) shows two numerically solutions of (2) with two close initial conditions ($u_1(0) = 1, u_2(0) = 2, u_3(0) = 3, u_4(0) = 4, u_5(0) = -5$ and $\tilde{u}_1(0) = 0.999, \tilde{u}_2(0) = 2, \tilde{u}_3(0) = 3.001, \tilde{u}_4(0) = 4, \tilde{u}_5(0) = -5$) (we plot only (t, u_1) diagram) and with the above choice of a, b and c. It is clear that from Fig. (1b) our system displays sensitive dependence on initial conditions.

(2.5) Dynamics of the new chaotic complex system:

The values of the parameters and the corresponding dynamical behaviors of (1) can be classified numerically for $a = 42, b = 6,$ as follows:
$(i) 0 < c < 1$, solutions of (1) approach the trivial fixed point $(0,0,0,0,0)$,
$(ii) 1 < c \leqslant 24.4,$ solutions of (1) approach one of the nontrivial fixed points
$(iii) 24.4 \leqslant c \leqslant 218$, system (1) has chaotic attractor, see Figures $1c$ and $1d$.
As shown in Figures $1c$ and $1d$, the system (1) exhibits chaotic attractor for $a = 42, b = 6$ and $c = 28$ with the initial conditions $(t = 0)$: $u_1(0) = 1, u_2(0) = 2, u_3(0) = 3, u_4(0) = 4, u_5(0) = -5.$ In Figure $1c$ we plot the 3-dimensional space (u_1, u_2, u_5), while in Figure $1d$ we plot the motion in the (u_3, u_4, u_5) space.

3-Nonlinear control technique.

Let us now study chaos synchronization in the complex system (2) using the idea of nonlinear control technique as follows: We assume that we have two complex systems and denote the drive system by the subscript 1, while the response system to be controlled is denoted by the subscript 2. The drive and response systems are thus defined respectively as:

$$\begin{aligned} \dot{x}_1 &= a(y_1 - x_1) + y_1 z_1, \\ \dot{y}_1 &= c x_1 - y_1 - x_1 z_1, \\ \dot{z} &= 1/2(\bar{x}_1 y_1 + x_1 \bar{y}_1) - b z_1, \end{aligned} \quad (9)$$

and

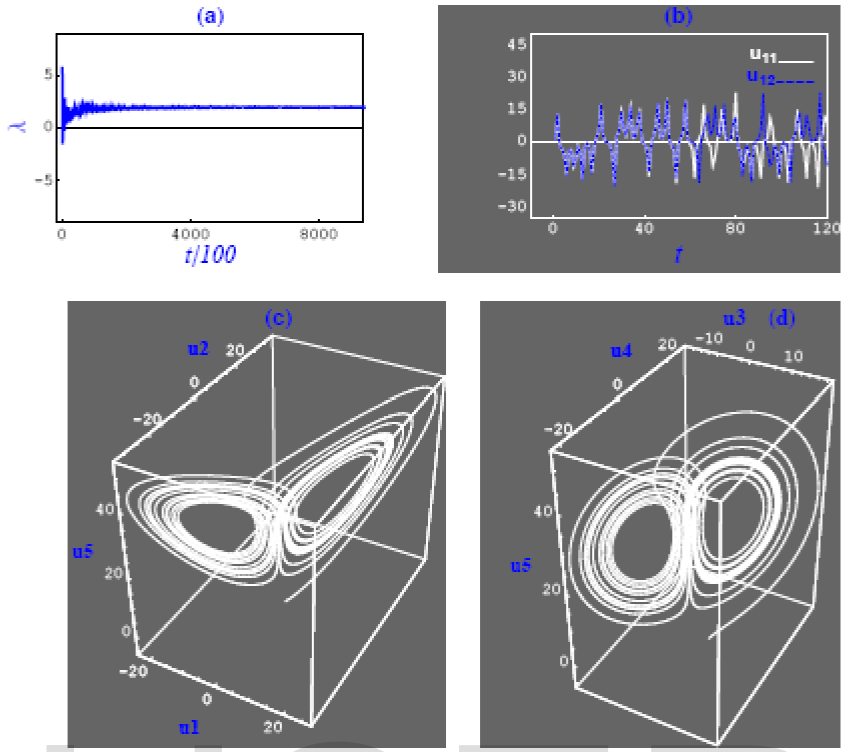

Figure 1: (a) The Maximum Lyapunov Exponent of this attractor is clearly positive, indicating that the motion on the attractor is chaotic for $a=42, b=6$ and $c=28$ with $t_0=0, u_1(0)=1, u_2(0)=2, u_3(0)=3, u_4(0)=4, u_5(0)=-5$ (b)Two numerically solutions of (2) for $a=42, b=6$ and $c=28$ with $t_0=0, u_1(0)=1, u_2(0)=2, u_3(0)=3, u_4(0)=4, u_5(0)=-5$ (the solid curve) and $u_1(0)=.999, u_2(0)=2, u_3(0)=3.001, u_4(0)=4, u_5(0)=-5$ (the dotted curve) , ($t=$ time/ 10).(c) A chaotic attractor of (3) at $a=42, b=6$ and $c=28$ in (u_1, u_3, u_5) space and the same value of initial conditions for the solid curve in (a).(d) A chaotic attractor of (3) in (u_2, u_4, u_5) space for the same values of parameters and initial conditions as in (a).

$$\begin{aligned}
\dot{x}_2 &= a(y_2 - x_2) + y_2 z_2 + (v_1 + i v_2), \\
\dot{y}_2 &= c x_2 - y_2 - x_2 z_2 + (v_3 + i v_4), \\
\dot{z}_2 &= 1/2(\bar{x}_2 y_2 + x_2 \bar{y}_2) - b z_2 + v_5,
\end{aligned} \qquad (10)$$

where $x_1 = u_{11} + i u_{21}, y_1 = u_{31} + i u_{41}$, $z_1 = u_{51}$, $x_2 = u_{12} + i u_{22}, y_2 = u_{32} + i u_{42}$ and $z_2 = u_{52}$, overbar denotes complex conjugation, $v_1 + i v_2$, $v_3 + i v_4$ and v_5 are complex and real control functions respectively, which are to be determined and all u_{ij} and v_i variables are real. The complex system (9) can be rewritten in the form of five real first order ODEs of the form:

$$\begin{aligned}
\dot{u}_{11} &= a(u_{31} - u_{11}) + u_{31}u_{51},\\
\dot{u}_{21} &= a(u_{41} - u_{21}) + u_{41}u_{51},\\
\dot{u}_{31} &= cu_{11} - u_{11}u_{51} - u_{31},\\
\dot{u}_{41} &= cu_{21} - u_{21}u_{51} - u_{41},\\
\dot{u}_{51} &= u_{11}u_{31} + u_{21}u_{41} - bu_{51}.
\end{aligned} \tag{11}$$

And the response system (10) in the real form:

$$\begin{aligned}
\dot{u}_{12} &= a(u_{32} - u_{12}) + u_{32}u_{52} + v_1,\\
\dot{u}_{22} &= a(u_{42} - u_{22}) + u_{42}u_{52} + v_2,\\
\dot{u}_{32} &= cu_{12} - u_{12}u_{52} - u_{32} + v_3,\\
\dot{u}_{42} &= cu_{22} - u_{22}u_{52} - u_{42} + v_4,\\
\dot{u}_{52} &= u_{12}u_{32} + u_{22}u_{42} - bu_{52} + v_5.
\end{aligned} \tag{12}$$

In order to obtain the control signals, we define as the errors between the drive and the response system to be controlled the quantities:

$$\begin{aligned}
e_{u_1} + ie_{u_2} &= x_2 - x_1 = (u_{12} - u_{11}) + i(u_{22} - u_{21}),\\
e_{u_3} + ie_{u_4} &= y_2 - y_2 = (u_{32} - u_{31}) + i(u_{42} - u_{41}),\\
e_{u_5} &= z_2 - z_1 = u_{52} - u_{51}.
\end{aligned} \tag{13}$$

and using

$$\begin{aligned}
u_{11}u_{51} - u_{12}u_{52} &= -u_{51}e_{u_1} - e_{u_5}u_{12},\\
u_{21}u_{51} - u_{22}u_{52} &= -u_{51}e_{u_2} - e_{u_5}u_{22},\\
u_{32}u_{52} - u_{31}u_{51} &= u_{32}e_{u_5} + e_{u_3}u_{51},\\
u_{42}u_{52} - u_{41}u_{51} &= u_{42}e_{u_5} + e_{u_4}u_{51},\\
u_{32}u_{12} - u_{11}u_{31} &= u_{32}e_{u_1} + e_{u_3}u_{11},\\
u_{42}u_{22} - u_{21}u_{41} &= u_{42}e_{u_2} + e_{u_4}u_{21}.
\end{aligned} \tag{14}$$

Subtracting (9) from (10) using (13) and (14) we get:

$$\begin{aligned}
\dot{e}_{u_1} + i\dot{e}_{u_2} &= [a(e_{u_3} - e_{u_1}) + u_{32}e_{u_5} + u_{51}e_{u_3}] \\
&\quad + i[a(e_{u_4} - e_{u_2}) + u_{42}e_{u_5} + u_{51}e_{u_4}] \\
&\quad + (v_1 + iv_2), \\
\dot{e}_{u_3} + i\dot{e}_{u_4} &= [ce_{u_1} - e_{u_3} - u_{51}e_{u_1} - u_{12}e_{u_5}] \\
&\quad + i[ce_{u_2} - e_{u_4} - u_{51}e_{u_2} - u_{22}e_{u_5}] \\
&\quad + (v_3 + iv_4), \\
\dot{e}_{u_5} &= u_{11}e_{u_3} + u_{32}e_{u_1} + u_{21}e_{u_4} + u_{42}e_{u_2} - be_{u_5} + v_5.
\end{aligned} \tag{15}$$

Equation (15) describes a dynamical system via which the errors evolve in time and finally the ODEs of this system, separating real from imaginary parts, become:

$$\begin{aligned}
\dot{e}_{u_1} &= a(e_{u_3} - e_{u_1}) + u_{32}e_{u_5} + u_{51}e_{u_3} + v_1, \\
\dot{e}_{u_2} &= (e_{u_4} - e_{u_2}) + u_{42}e_{u_5} + u_{51}e_{u_4} + v_2, \\
\dot{e}_{u_3} &= ce_{u_1} - e_{u_3} - u_{51}e_{u_1} + u_{12}e_{u_5} + v_3, \\
\dot{e}_{u_4} &= ce_{u_2} - e_{u_4} - u_{51}e_{u_2} + u_{22}e_{u_5} + v_4, \\
\dot{e}_{u_5} &= u_{11}e_{u_3} + u_{32}e_{u_1} + u_{21}e_{u_4} + u_{42}e_{u_2} - be_{u_5} + v_5.
\end{aligned} \tag{16}$$

For positive parameters a, b and c, we may now define a Lyapunov function by the following quantity:

$$V(t) = 1/2 \sum_{i=1}^{5} e_{u_i}^2. \tag{17}$$

The derivative of $V(t)$ along the solution of system (15) is:

$$\begin{aligned}
\dot{V}(t) &= -(ae_{u_1}^2 + ae_{u_2}^2 + e_{u_3}^2 + e_{u_4}^2 + be_{u_5}^2) \\
&\quad + e_{u_1}[ae_{u_3} + u_{32}e_{u_5} + u_{51}e_{u_3}] \\
&\quad + e_{u_1}[ae_{u_4} + u_{42}e_{u_5} + u_{51}e_{u_4}] \\
&\quad + e_{u_3}[ce_{u_1} - e_{u_5}u_{12} - u_{51}e_{u_1}] \\
&\quad + e_{u_4}[ce_{u_2} - e_{u_5}u_{22} - u_{51}e_{u_2}] \\
&\quad + e_{u_5}[u_{11}e_{u_3} + u_{32}e_{u_1} + u_{21}e_{u_4} + u_{42}e_{u_2}] + \textstyle\sum_{i=1}^{5} v_i e_{u_i}.
\end{aligned} \tag{18}$$

If we choose the active control functions v_i such that:

$$\begin{aligned} v_1 &= (a-1)e_{u_1} - ae_{u_3} - u_{32}e_{u_5} - u_{51}e_{u_3}, \\ v_2 &= (a-1)e_{u_2} - ae_{u_4} - u_{42}e_{u_5} - u_{51}e_{u_4}, \\ v_3 &= -ce_{u_1} + u_{51}e_{u_1} + u_{12}e_{u_5}, \\ v_4 &= -ce_{u_2} + u_{51}e_{u_2} + u_{22}e_{u_5}, \\ v_5 &= (b-1)e_{u_5} - [u_{11}e_{u_3} + u_{32}e_{u_1} + u_{21}e_{u_4} + u_{42}e_{u_2}]. \end{aligned} \tag{19}$$

equation (18) yields:

$$\dot{V}(t) = -(e_{u_1}^2 + e_{u_2}^2 + e_{u_3}^2 + e_{u_4}^2 + e_{u_5}^2) < 0. \tag{20}$$

Since $V(t)$ is a positive definite function and its derivative is negative definite, then Lyapunov's direct method implies that the equilibrium point $e_{u_i} = 0, i = 1,2,3,4,5$ of the system (16) is asymptotically stable, which means that $e_{u_i} \to 0$ as $t \to \infty$, $i = 1,2,\ldots,5$. Systems (9) and (10) with (19) are solved numerically (using e.g. Mathematica 5.2 software), for $a = 42, b = 6$ and $c = 28$ with the initial conditions $u_{11}(0) = 1, u_{21}(0) = 2, u_{31}(0) = 3, u_{41}(0) = 4, u_{51}(0) = -5$ and $u_{12}(0) = -11, u_{22}(0) = -30, u_{32}(0) = -18, u_{42}(0) = -17, u_{52}(0) = 45.$
The simulation results are illustrated in Figures 2 and 3 . In Figure 2 the solutions of (9) and (10) are plotted subject to different initial conditions and show that the chaos synchronization is achieved after very small values of t . In Figure 3 it can be seen that the synchronization errors e_{u_i} converge to zero, as expected from the above analytical considerations.

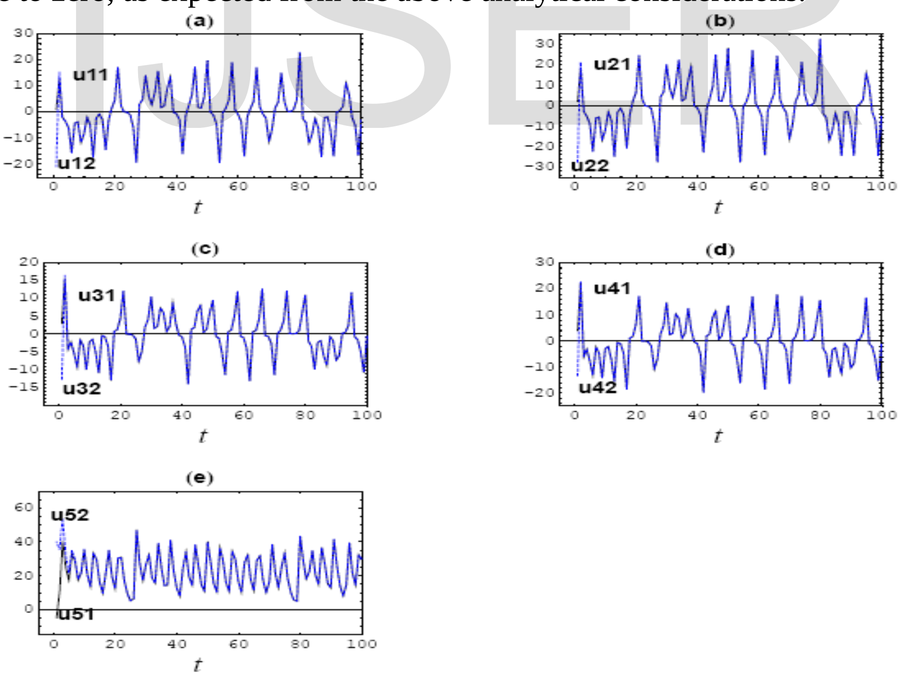

Figure 2: Chaos synchronization of systems (4.1) and (5.1) with (12) for $a = 42, b = 6$ and $c = 28$ with $t_0 = 0, u_{11}(0) = 1, u_{21}(0) = 2, u_{31}(0) = 3, u_{41}(0) = 4, u_{51}(0) = -5$ and $u_{12}(0) = -11, u_{22}(0) = -30, u_{32}(0) = -18, u_{42}(0) = -17$, $u_{52}(0) = 45$. (a) $u_{11}(t)$ and $u_{12}(t)$ versus t, (b) $u_{21}(t)$ and $u_{22}(t)$ versus t, (c) $u_{31}(t)$ and $u_{32}(t)$ versus t, (d) $u_{41}(t)$ and $u_{42}(t)$ versus t, (e) $u_{51}(t)$ and $u_{52}(t)$ versus t ($t =$ time/ 10).

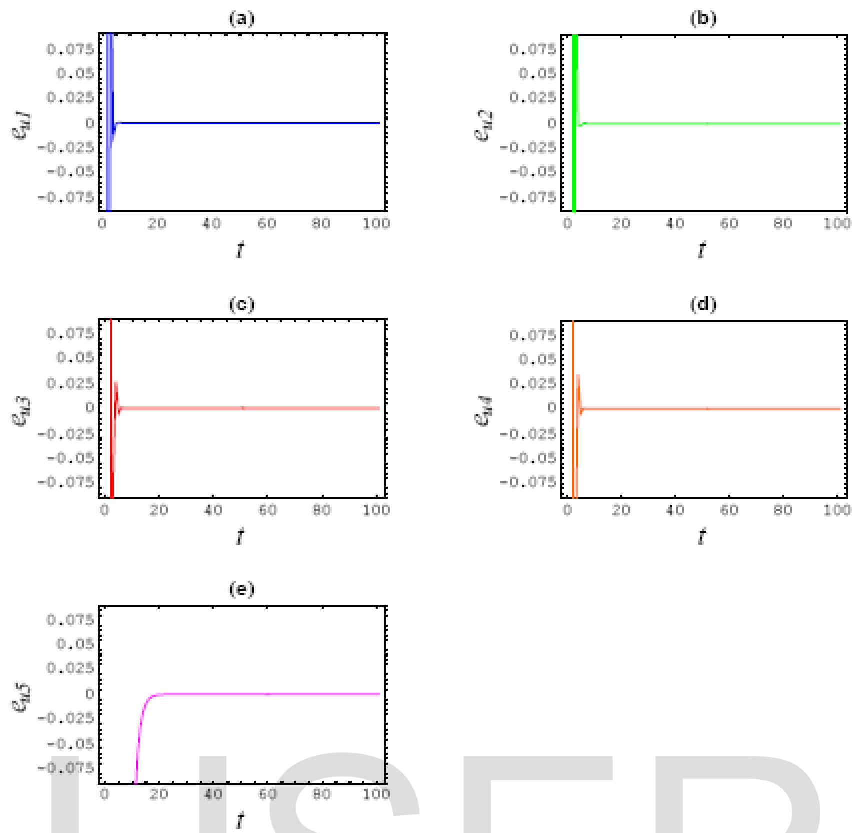

Figure 3: Synchronization errors (solutions of system (16))
(a) (e_{u_1}, t) diagram, (b) (e_{u_2}, t) diagram, (c) (e_{u_3}, t) diagram, (d) (e_{u_4}, t) diagram, (e) (e_{u_5}, t) diagram.

Conclusions.

In this paper, we have introduced a new chaotic complex nonlinear system (2). This system with real variables has been introduced and studied in the recent years. Our new complex system appears in many important applications in engineering, for example, in communications where doubling the number of variables (i.e. introducing complex variables) may be used to increase the content and security of the transmitted information. The basic properties of system (2) including invariance, dissipativity, equilibria and their stability, Lyapunov exponents, chaotic behavior and chaotic attractors are studied. The chaos synchronization of this chaotic attractors are studied via nonlinear control technique as one case of the values of parameters which generate chaos. Other cases of these parameters can be similarly treated. A Lyapunov function is derived to prove that the error system is asymptotically stable. The results of chaos synchronization and the error, are shown in Figures 2 and 3.

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